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On a Small Perturbation in the Two Dimensional Inverse Conductivity Problem

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1. INTRODUCTION

Let D be a simply connected domain with $C^{1,1}$ boundary ∂D in the complex plane $\mathbb{C}$. Consider the refraction problem

$$\begin{aligned} \operatorname{div}[(1+\chi(D))\nabla u] &= F \qquad \text{in } \mathbb{C} \\ u(\infty) &= 0. \end{aligned} \tag{1.1}$$

Here $\chi(D)$ is the indicator function of D; $F(z)=\delta(z-a)-\delta(z-b)$, where δ is Dirac's delta function; and a and b are distinct elements of $\mathbb{C}\backslash\bar{D}$. The existence and uniqueness of the solution u to (1.1) are covered in Section 3.

With the notation

$$D^e=\mathbb{C}\backslash\bar{D}, \qquad D^i=D, \qquad u^e=u|_{D^e}, \qquad u^i=u|_{D^i},$$

u^e, u^i, and their first derivatives are Hölder continuous in $\bar{D}^e\backslash\{a,b\}$ and $\bar{D}^i$, respectively; see [6, p. 222]. The solution satisfies

$$\Delta u^e = F \text{ ind } D^e, \qquad \Delta u^i = 0 \text{ in } D^i \tag{1.2}$$

$$u^e=u^i, \qquad \frac{\partial u^e}{\partial N}=2\frac{\partial u^i}{\partial N}, \qquad \frac{\partial u^e}{\partial T}=\frac{\partial u^i}{\partial T} \qquad \text{on } \partial D, \tag{1.3}$$

where N and T respectively denote the outward normal to ∂D and the tangent to ∂D obtained by rotating N counterclockwise $\pi/2$ radians.

Assume that D_0 is a known bounded domain with analytic boundary ∂D_0, and u_0 is the solution to (1.1) with $D=D_0$ and a, b outside the closure of some disk B which contains $\bar{D}_0$. We study the following inverse problem. Suppose that a function u^e satisfying $\Delta u^e=F$ in $\mathbb{C}\backslash\bar{B}$ and $u^e(\infty)=0$ is "close" in $\mathbb{C}\backslash\bar{B}$ to u_0^e and has a harmonic continuation to

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a neighbourhood of $B \setminus D_0$. Can one find a domain D close to D_0 and a function u^i on D such that u^e, u^i, and D satisfy the refraction conditions (1.2), (1.3)?

In a bounded domain Ω with $C^{2,\alpha}$ boundary $\partial\Omega$, $\bar{D} \subset \Omega$, we consider the problem

$$\operatorname{div}[(1+\chi(D))\nabla u] = 0 \text{ in } \Omega, \qquad \frac{\partial u}{\partial N} = g \text{ on } \partial\Omega, \tag{1.4}$$

where g is in $C^{1,\alpha}(\partial\Omega)$, not identically zero, and

$$\int_{\partial\Omega} g = 0.$$

With u normalized by

$$\int_{\partial\Omega} u = 0$$

this problem (see [6]) has a unique solution u in $H^1(\Omega) \cap C^\alpha(\bar{\Omega})$ for some $\alpha > 0$, with second derivatives of u^e continuous up to $\partial\Omega$. The solution satisfies (1.3) and

$$\Delta u^e = 0 \text{ in } D^e = \Omega \setminus \bar{D}, \qquad \Delta u^i = 0 \text{ in } D^i = D. \tag{1.5}$$

For the inverse problem in Ω, suppose D_0 with $\bar{D}_0 \subset \Omega$, ∂D_0 analytic, and u_0, the solution to (1.4) with $D = D_0$, are known, with g prescribed as follows. Let z_1, z_2 be on $\partial\Omega$, with $z_1 \neq z_2$ and Ω strictly convex near the z_j. Parametrized counterclockwise, z on $\partial\Omega$ is $z = z(\tau)$, $\tau \in [0, 1]$, with $z_1 = z(0) = z(1)$ and $z_2 = z(\sigma)$ for some σ in $(0, 1)$. Choose

$$\begin{aligned} &g > 0 \text{ on } (0, \sigma), \qquad g < 0 \text{ on } (\sigma, 1) \\ &g'(0) > 0, \qquad g'(\sigma) < 0. \end{aligned} \tag{1.6}$$

The datum u^e is assumed to satisfy $\partial u^e/\partial N = g$ on $\partial\Omega$ and the normalization condition, and to have a harmonic continuation to a neighborhood of $\bar{D}_0^e$. A slight modification of the discussion in Sections 4 and 5 of the inverse problem in the whole plane yields the same result in Ω.

Uniqueness and solvability will be seen to depend on the index of the gradient u_{0z}^e over ∂D_0, the index of a continuous vector field $V = a + ib$ over a closed, piecewise smooth contour Γ being defined as

$$\kappa(V; \Gamma) = \frac{1}{2\pi}[\arg(a - ib)]_\Gamma,$$

provided $V \neq 0$ on Γ. Bellout *et al.* used index theory for an inverse problem in [1].

We remark that the following result is local. There are few global uniqueness results in the inverse conductivity problem. Only for disks, convex polyhedrons, and cylinders has uniqueness been proved; see [3, 5]. Related results in the inverse problem of potential theory (see [4]) are also local, and there exist counterexamples to global solvability in the inverse potential problem.

2. Statement of Results

Let $z_0(t)$ be the conformal map of the unit disk to D_0 which satisfies

$$z_0(0) = 0, \qquad z_0'(0) > 0, \tag{2.1}$$

where D_0 is a bounded, simply connected domain with analytic boundary ∂D_0. Let u_0 be the solution to (1.1) with $D = D_0$ and a, b outside the closure of some disk B which contains $\bar{D}_0$, or the solution to (1.4), (1.6) with $D = D_0$.

Suppose u^e is a datum which satisfies

$$\Delta u^e = F \text{ on } \mathbb{C} \setminus \bar{B}, \qquad u^e(\infty) = 0,$$

and which has a harmonic continuation to a neighborhood of $B \setminus D_0$ with prescribed bound in $C^{3+\lambda}(B \setminus D_0)$.

THEOREM 2.1. *There are numbers* $\varepsilon_1, \varepsilon_2 > 0$ *such that if*

$$|u^e - u_0^e|_{3+\lambda} (\mathbb{C} \setminus \bar{B}) < \varepsilon_1$$

then at most one pair u^i, ω *satisfies* (i) $|\omega|_{1+\lambda} (|t| \leqslant 1) < \varepsilon_2$, (ii) ω *is analytic in* $|t| < 1$,

$$\text{(iii)} \qquad \omega(0) = 0, \qquad \operatorname{Im} \omega'(0) = 0, \tag{2.2}$$

(iv) u^i *is harmonic in* $D = \{z \in \mathbb{C} \mid z = z_0(t) + \omega(t), |t| < 1\}$, *and* (v) D *in* (iv), u^i, *and* u^e *satisfy* (1.2), (1.3). *In addition, the existence of* u^i, ω *is assured by fulfillment of the orthogonality condition* (7.4), *which defined in Section* 7.

For the inverse problem (1.4) suppose u^e satisfies, with g as in (1.6),

$$\Delta u^e = 0 \text{ near } \bar{D}_0^e, \qquad \frac{\partial u^e}{\partial N} = g \text{ on } \partial\Omega,$$

$$\int_{\partial\Omega} u^e = 0.$$

THEOREM 2.2. *There are numbers* $\varepsilon_1, \varepsilon_2 > 0$ *such that if*

$$|u^e - u_0^e|_{3+\lambda}(\partial\Omega^e) < \varepsilon_1$$

then at most one pair u^i, ω *satisfies* (i) *through* (iv) *in Theorem* 2.1, *and* (v) D, u^i, *and* u^e *satisfy* (1.3), (1.5). *Existence of* u^i, ω *depends on the condition* (7.4).

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION TO THE DIRECT PROBLEM

THEOREM 3.1. *There is a unique solution of problem* (1.1).

Proof. The unique solution to $\Delta u = F$ in $\mathbb{C}$ satisfying $u(\infty) = 0$ is

$$u_1(z) = \ln|z-a| - \ln|z-b|.$$

Let $u = u_1 + v$, where v satisfies

$$\Delta v^i = 0 \text{ in } D^i, \qquad \Delta v^e = 0 \text{ in } D^e$$

$$v^e = v^i, \qquad \frac{\partial v^e}{\partial N} - 2\frac{\partial v^i}{\partial N} = \frac{\partial u_1}{\partial N} \qquad \text{on } \partial D \tag{3.1}$$

$$v^e(\infty) = 0. \tag{3.2}$$

One seeks v of the form

$$v(x) = \int_{\partial D} g(y) \ln|x-y| \, d\Gamma(y).$$

By the jump relations [4, Theorem 1.6.2], the second boundary condition in (3.1) is equivalent to

$$g(x) + \frac{2}{3}\frac{\partial}{\partial N}\int_{\partial D} g(y)\ln|x-y| \, d\Gamma(y) = -\frac{2}{3}\frac{\partial u_1}{\partial N}(x). \tag{3.3}$$

The integral in (3.3) is a compact operator on $C(\partial D)$, by [4, Lemma 1.6.5]. To show invertibility of the operator in (3.3), suppose g satisfies (3.3) with zero right side. Then $v(x)$, by [4, Lemma 1.5.6], satisfies

$$v^e(x) = C\ln|x| + v_0(x), \qquad v_0(\infty) = 0,$$

with

$$C = \int_{\partial D} g(y) \, d\Gamma(y). \tag{3.4}$$

But the jump relations and the second boundary condition in (3.1) imply that $g = -\partial v^i/\partial N$, with $\Delta v^i = 0$ in D^i. Hence, $C = 0$; so $v^e(\infty) = 0$ and Giraud's principle with (3.1) imply that v is identically zero; so the jump relations show, that $g(y)$ is zero. So the operator in (3.3) is invertible in $C(\partial D)$, and there is a g in $C(\partial D)$ that satisfies (3.3), hence a v that satisfies (3.1). The jump relations and (3.1) show that $C = 0$ in (3.4), so $v^e(\infty) = 0$.

Now Giraud's principle can be used to show that $u = u_1 + v$ uniquely satisfies (1.2), (1.3), which proves the theorem.

4. The Linearized Problem

Let u^e, u^i satisfy (1.1), (1.2), (1.3) for some D. Let G be a bounded, simply connected region with

$$a, b \notin G \qquad \text{and} \qquad \bar{D} \subset G.$$

Using (1.2), (1.3) to show that the conjugate period of u^e over ∂D is zero, we define in G, uniquely up to arbitrary additive imaginary constants,

$$U^e = u^e + iv^e, \qquad U^i = u^i + iv^i, \tag{4.1}$$

v^e, v^i being the conjugate harmonic functions.

We use the technique of [2] to obtain a boundary condition for U^e, U^i which reflects (1.3). The Cauchy–Riemann conditions and (1.3) imply that

$$\frac{\partial v^e}{\partial T} - 2\frac{\partial v^i}{\partial T} = 0;$$

so, fixing v^e by choosing a point $z_1 \in \partial D$ and setting $v^e(z_1) = 0$, we have for some real constant ξ

$$v^e = 2v^i + \xi \qquad \text{on } \partial D. \tag{4.2}$$

From (1.3) and (4.2) we obtain on ∂D

$$U^e + \bar{U}^e = U^i + \bar{U}^i \tag{4.3}$$

$$(U^e - \bar{U}^e) - 2(U^i - \bar{U}^i) = 2i\xi. \tag{4.4}$$

Substituting $\bar{U}^i$ from (4.3) into (4.4) results in

$$4U^i(z) = 3U^e(z) + \overline{U^e(z)} - 2i\xi, \qquad z \in \partial D. \tag{4.5}$$

If $z(t)$ is the conformal map of the unit disk onto D which satisfies

$$z(0) = 0, \qquad z'(0) > 0, \tag{4.6}$$

then (4.5) becomes

$$4U^i(z(t)) = 3U^e(z(t)) + \overline{U^e(z(t))} - 2i\xi, \qquad |t| = 1. \tag{4.7}$$

We show that the condition (4.7) is equivalent to the inverse problem. Let u be the data in the inverse problem; i.e., u satisfies $\Delta u = F$ outside some ball B centered at 0, $u(\infty) = 0$, and u has a harmonic continuation into B. Set $U = u + iv$, with v the harmonic conjugate of u. To see that v exists in a region suited to our purpose, consider any simply connected domain D with $C^{1,1}$ boundary such that $B \backslash \bar{D}$ is contained in the region where u is defined. Solve the direct problem for D to get unique u^e; from the preceding discussion, the conjugate period of u^e is zero over ∂D. Let $\tilde{u} = u^e - u$. Then $\Delta \tilde{u} = 0$ in simply connected $D^e \cup \{\infty\}$, so there is a single-valued harmonic conjugate $\tilde{v}$ of $\tilde{u}$ in D^e. So $v = v^e - \tilde{v}|_{B \backslash D}$ is the harmonic conjugate of u in $B \backslash D$, uniquely determined by, say, $v(z_2) = 0$ for some z_2 on ∂D.

THEOREM 4.1. *Let $\phi^+(t)$, $z(t)$ be analytic in $|t| < 1$, z one-sheeted, ϕ^+ and z in $C^1(|t| \leqslant 1)$, and z satisfy* (4.6). *If*

$$\phi^+(t) = 3U(z(t)) + \overline{U(z(t))} - 2i\xi, \qquad |t| = 1, \tag{4.8}$$

then the pair $\frac{1}{4}\operatorname{Re}\phi^+(t)$, $D = \{z : z = z(t), |t| < 1\}$ is a solution to the inverse problem.

Proof. Solve the direct problem for D to obtain (4.7). Let $\tilde{v}$ be the conjugate harmonic function of $\tilde{u} = u^e - u$ in D^e with $\tilde{v}(\infty) = 0$. On $B \backslash D$, $\tilde{v} = v^e - v + \eta$, η a real constant. Subtract $2i\eta$ from both sides of (4.8); then subtract (4.8) from (4.7) and use the inverse $t = t(z)$ of $z(t)$ to get

$$\psi^i(z) = 3\psi^e(z) + \overline{\psi^e(z)} \text{ on } \partial D, \qquad \psi^e(\infty) = 0, \tag{4.9}$$

where $\psi^i = 4U^i - \phi^+(t) + 2i\eta$, $\psi^e = U^e - U + i\eta$, ψ^i is analytic in D^i, and ψ^e is analytic in $D^e \cap B$ with analytic continuation to D^e given by $\psi^e = \tilde{u} + i\tilde{v}$, $\tilde{v}(\infty) = 0$. If $\psi = \alpha + i\beta$, then (4.9), differentiation in the tangent direction, and the Cauchy–Riemann conditions show that

$$\frac{\partial \beta^i}{\partial N} - 4\frac{\partial \beta^e}{\partial N} = 0, \qquad \frac{\partial \alpha^i}{\partial N} - 2\frac{\partial \alpha^e}{\partial N} = 0 \text{ on } \partial D,$$

so Giraud's principle with $\alpha(\infty) = 0$, $\beta(\infty) = 0$ can be applied to show that (4.9) has only the trivial solution. Thus, $u^e = u$; so the pair $\frac{1}{4}\operatorname{Re}\phi^+(t(z))$, D is a solution of the inverse problem. The proof is complete.

Remark. The equivalence is independent of choice of constants $v^e(z_1)$, $v(z_2)$, ξ.

For the domain Ω, suppose D is any domain with $C^{1,1}$ boundary such that $\Omega \backslash D$ is in the region where the data is defined. Solve (1.4), (1.6) for D. Then use Green's formula and the fact that $\partial \tilde{u}/\partial N = 0$ on $\partial\Omega$ to show that the conjugate period of $\tilde{u}$ over ∂D is zero, hence that of $u = u^e - \tilde{u}$ is zero. Set $U = u + iv$ and fix v by setting $v(z_2) = 0$ for some z_2 on $\partial\Omega$.

Proof of Theorem 4.1 *for the Problem in* Ω. Solve the direct problem for D to get (4.7). Subtract (4.8) from (4.7) and use the inverse $t(z)$ to get

$$\psi^i(z) = 3\psi^e(z) + \overline{\psi^e(z)} \text{ on } \partial D, \qquad \frac{\partial \alpha^e}{\partial N} = 0 \text{ on } \partial\Omega,$$

with $\psi^i = 4U^i - \phi^+(t)$ analytic in D^i, $\psi^e = U^e - U$ analytic in D^e, and $\psi = \alpha + i\beta$. The Cauchy–Riemann condition implies that $\partial\beta^e/\partial T = 0$ on $\partial\Omega$; hence, β^e is constant on $\partial\Omega$. So the fact that

$$\frac{\partial \beta^i}{\partial N} - 4\frac{\partial \beta^e}{\partial N} = 0 \qquad \text{on } \partial D$$

and Giraud's principle imply β is constant on D^e and D^i; so α is constant on those regions, too. Hence, $\alpha^e = u^e - u$ is constant. So due to the normalization conditions on u^e, u over $\partial\Omega$, $\alpha = 0$. It follows that the pair $\frac{1}{4}\operatorname{Re}\phi^+(t(z))$, D is a solution of the inverse problem, which finishes the proof.

Now let $z_0(t)$ be the conformal map of the unit disk to D_0 which satisfies (2.1). We are looking for a domain D whose conformal map from the unit disk is

$$z(t) = z_0(t) + \omega(t),$$

where ω is some function analytic in $|t| < 1$ that satisfies (2.2) and is small enough for z to satisfy (4.6).

Suppose B is a disk such that $\bar{D}_0 \subset B$ and $a, b \notin \bar{B}$, with datum u^e for the inverse problem given outside of B and having a harmonic continuation to a neighborhood of $B \backslash D_0$. In B, let U_0^e, U_0^i be the functions in (4.1) for $D = D_0$, with v_0^e fixed by choosing a points z_0 on ∂D_0 and setting $v_0^e(z_0) = 0$. Let $U^e = u^e + iv^e$, where $v^e = v_0^e - \tilde{v}|_B$, $\tilde{v}$ being the harmonic conjugate of $\tilde{u} = u_0^e - u^e$ satisfying $\tilde{v}(z_1) = 0$ for some z_1 in ∂B or $\partial\Omega$.

For the purpose of finding ϕ^+, ω analytic in $|t| < 1$, let

$$\phi^+(t) = 3U^e(z_0(t) + \omega(t)) + \overline{U^e(z_0(t) + \omega(t))} - 2i\xi \qquad \text{on} \quad |t| = 1. \tag{4.10}$$

Add and subtract

$$3U_0^e(z_0(t)) + \overline{U_0^e(z_0(t))} + 3u_{0z}^e(z_0(t))\,\omega(t) + \overline{u_{0z}^e(z_0(t))\,\omega(t)}$$

to (4.10) to obtain

$$\begin{aligned} \phi_1^+(t) &= 3u_{0z}^e(z_0(t))\,\omega(t) + \overline{u_{0z}^e(z_0(t))\,\omega(t)} + B\omega(t) \qquad \text{on} \quad |t| = 1 \\ \phi_1^+(t) &= \phi^+(t) - 4U_0^i(z_0(t)) + 2i\xi \\ B\omega &= 3B_1\omega + \overline{B_1\omega} \\ B_1\omega &= U^e(z_0+\omega) - U_0^e(z_0) - u_{0z}^e(z_0)\omega. \end{aligned} \tag{4.11}$$

Taking the tangential derivatives of both sides of

$$4U_0^i(z_0(t)) = 3U_0^e(z_0(t)) + \overline{U_0^e(z_0(t))} \qquad \text{on} \quad |t| = 1$$

results in

$$3u_{0z}^e(z_0(t)) = 4u_{0z}^i(z_0(t)) + \overline{u_{0z}^e(z_0(t))\, z_0'(t) t} / z_0'(t) t,$$

which substituted into (4.11) yields on $|t| = 1$

$$\begin{aligned} \phi(t) &= [\overline{u_{0z}^e(z_0(t))\, z_0'(t) t} / z_0'(t) t]\,\omega(t) + \overline{u_{0z}^e(z_0(t))\,\omega(t)} + B\omega(t) \\ \phi(t) &= \phi_1^+(t) - 4u_{0z}^i(z_0(t))\,\omega(t). \end{aligned} \tag{4.12}$$

Using $\omega_*(t) = \overline{\omega(1/\bar{t})}$, (4.12) becomes the following boundary value problem of a type considered by Mikhailov [7, pp. 188–193].

Find ω_* analytic in $|t| > 1$ and ϕ analytic in $|t| < 1$ such that

$$\phi(t) = \overline{u_{0z}^e(z_0(t))}\,\omega_*(t) + [\overline{u_{0z}^e(z_0(t))\, z_0'(t) t} / z_0'(t) t]\,\overline{\omega_*(t)} + B\omega(t) \tag{4.13}$$

on $|t| = 1$.

Equation (4.13) defines an operator $A(\phi, \omega) = B\omega$ from a space $Y \times Z$, where Y is the set of functions analytic in $|t| < 1$ which are in $C^{1+\lambda}$ ($|t| \leqslant 1$), and Z is the space of functions analytic in $|t| < 1$ which are in a ball $|\omega|_{1+\lambda}$ ($|t| \leqslant 1) < \varepsilon_2$ for some $\varepsilon_2 > 0$ to be determined, and which satisfy (2.2), to the space $C^{1+\lambda}$ ($|t| = 1$). In order to investigate solvability and uniqueness in $Y \times Z$ of this problem, the coefficients of ω_* and $\overline{\omega_*}$ must first be shown to be nonzero on $|t| = 1$ and their indices κ must be determined.

5. $u_{0z}^e \neq 0$ ON ∂D_0 AND $\kappa(u_{0z}^e; \partial D_0) = 0$

Since ∂D_0 is analytic, solutions u_0^i, u_0^e to (1.1) or (1.4) with $D = D_0$ have harmonic continuations to a neighborhood of ∂D_0, so any zeroes on ∂D_0 of u_{0z}^i or u_{0z}^e are isolated. From condition (1.3) it can be shown that

$$u_{0z}^i(z) = 0 \text{ for } z \in \partial D_0 \qquad \text{if and only if} \quad u_{0z}^e(z) = 0. \tag{5.1}$$

Denote by $z_1, ..., z_l$ the zeros of u^i_{0z} on ∂D_0. (5.1) implies $u^e_{0z} \neq 0$ on $\partial D_0 \backslash \{z_1, ..., z_l\}$. Let L_ε be the Jordan curve formed by arcs of circles $\partial B(z_j; \varepsilon)$ centered at z_j which are contained in $\mathbb{C} \backslash D_0$ and

$$\Gamma_\varepsilon = \partial D_0 \Big\backslash \left[\bigcup_{j=1}^{l} B(z_j; \varepsilon) \right].$$

The following lemma and its proof are in [1].

LEMMA 5.1. *For some small $\varepsilon > 0$, the vector field u^e_{0z} on L_ε is homotopic to the vector field u^i_{0z} on L_ε.*

Also, recalling from Section 3 that $u_0 = u_1 + v$ with v^e harmonic in D^e and that u_{1z} has poles of order one at a and b, observe that u^e_{0z} has poles of order one at a and b.

THEOREM 5.1. *The gradient of the solution to* (1.1) *with $D = D_0$ vanishes nowhere but at ∞.*

Proof. Since the zero at ∞ is isolated, there exists a large disk B_1 with $a, b \in B_1$, $\bar{D}_0 \subset B_1$, which has the property that the only zero of u^e_{0z} lying outside of B_1 is at ∞. Let B_2, B_3 be disks centered at a, b with disjoint closures in B_1 disjoint from $\bar{D}_0$; in which a and b, respectively, are isolated poles of u^e_{0z}; and in which lie no zeros of u^e_{0z}. Let $B_4, ..., B_m$ be disks centered at and isolating the $m-3$ zeros of u^e_{0z} in $B_1 \backslash \bar{D}_0$, whose mutually disjoint closures are in B_1 disjoint from $\bar{D}_0$, $\bar{B}_2$, $\bar{B}_3$. Let ε be small enough so that D_ε, the domain bordered by L_ε, has closure in B_1 disjoint from all of the $\bar{B}_j$, $j > 1$. Finally, define the domain Ω by

$$\Omega = B_1 \Big\backslash \left[\bigcup_{j=2}^{m} \bar{B}_j \cup \bar{D}_\varepsilon \right].$$

Observe that the condition $u^e(\infty) = 0$ implies that the zero of u^e_{0z} at ∞ has order of at least 2. Hence, Lemma 5.1 and the observation following Lemma 5.1 yield

$$\begin{aligned} 0 = \kappa(u^e_{0z}; \partial\Omega) &= \kappa(u^e_{0z}; \partial B_1) - \kappa(u^e_{0z}; L_\varepsilon) - \sum_{j=2}^{m} \kappa(u^e_{0z}; \partial B_j) \\ &\leqslant -\kappa(u^i_{0z}; L_\varepsilon) - \sum_{j=4}^{m} \kappa(u^e_{0z}; \partial B_j). \end{aligned} \tag{5.2}$$

Hence, $\kappa(u^i_{0z}; L_\varepsilon) = 0$ and $\kappa(u^e_{0z}; \partial B_j) = 0$, $j = 4, ..., m$, which concludes the proof. In particular, (5.1) and Theorem 5.1 imply that $u^e_{0z} \neq 0$ on ∂D_0 and $\kappa(u^e_{0z}; \partial D_0) = 0$.

In the case of problem (1.4), let T_j, N_j, $j=1, 2$ be the unit tangent and normal to $\partial\Omega$ at z_j. Then (1.6) implies there are neighbourhoods V_j of z_j such that

$$\frac{\partial^2 u_0^e}{\partial T_1\, \partial N_1} > 0 \text{ in } V_1 \cap \Omega, \qquad \frac{\partial^2 u_0^e}{\partial T_2\, \partial N_2} < 0 \text{ in } V_2 \cap \Omega. \tag{5.3}$$

Choose straight lines l_j parallel to T_j so that $l_j \cap \Omega$ is in V_j, and let

$$l_1 \cap \partial\Omega = \{s_1, s_2\}, \qquad l_2 \cap \partial\Omega = \{s_3, s_4\},$$

so that $s_1, z_1, s_2, s_3, z_2, s_4$ are ordered counterclockwise on $\partial\Omega$; because the zeros of the gradient of u_0^e are isolated in $D^e = \Omega \setminus \bar{D}_0$, the l_j can be chosen so that $u_{0z}^e \neq 0$ on the intervals (s_1, s_2) and (s_3, s_4), and so that the domain Ω_1 with piecewise smooth boundary

$$\Gamma_1 = \{\partial\Omega \setminus [\operatorname{arc}(s_1, s_2) \cup \operatorname{arc}(s_3, s_4)]\} \cup \{[s_1, s_2] \cup [s_3, s_4]\}$$

has $\bar{D}_0 \subset \Omega_1$; arcs on $\partial\Omega$ are traversed counterclockwise. Note that $u_{0z}^e \neq 0$ on Γ_1, and note that (5.3) implies $\partial u_0^e/\partial N_1$ is strictly increasing on (s_1, s_2) and $\partial u_0^e/\partial N_2$ is strictly decreasing on (s_3, s_4).

LEMMA 5.2. *If Γ_1 is constructed as above, then $\kappa(u_{0z}^e; \Gamma_1) \leqslant 0$.*

Proof. Observe that (i) $\kappa(u_{0z}^e; \Gamma_1) > 0$ indicates a full clockwise turn of the vector ∇u_0^e as Γ_1 is traversed counterclockwise, and that (ii) (5.3) implies that ∇u_0^e points into Ω_1 from $\operatorname{arc}(s_4, s_1)$ and into the exterior of Ω_1 from $\operatorname{arc}(s_2, s_3)$. Condition (5.3) limits the behavior of ∇u_0^e on (s_1, s_2) to three cases: (1) ∇u_0^e points into Ω_1 from (s_1, s), crosses l_1 once clockwise, and points out from Ω_1 on (s, s_2). (2) ∇u_0^e points into Ω_1 from (s_1, s), crosses l_1 once counterclockwise, and traverses (s, s_2) pointing out from Ω_1. (3) ∇u_0^e traverses (s_1, s_2) without crossing l_1. On (s_3, s_4) there are also three cases: (1) ∇u_0^e points out from Ω_1 on (s_3, s), crosses l_2 once clockwise, and points into Ω_1 from (s, s_4). (2) ∇u_0^e points out from Ω_1 on (s_3, s), crosses l_2 once counterclockwise, and points into Ω_1 from (s, s_4). (3) ∇u_0^e traverses (s_3, s_4) without crossing l_2. Inspection of all nine combinations reveals that

$$\kappa(u_{0z}^e; \Gamma_1) \in \{-2, -1, 0\},$$

which proves the lemma.

THEOREM 5.2. *The gradient of the solution to* (1.4), (1.5) *is nowhere zero in Ω_1, $u_{0z}^e \neq 0$ on ∂D_0, and $\kappa(u_{0z}^e; \partial D_0) = 0$.*

Proof. The conclusion follows from a modification of the proof of Theorem 5.1.

6. $B\omega$ Is a Contraction Mapping

First observe from the uniqueness of continuation of harmonic functions that if $\varepsilon > 0$, then ε_1 in the hypothesis of Theorem 2.1 can be chosen small enough that

$$|u^e - u_0^e|_{3+\lambda}(N) < \varepsilon,$$

where N is a neighborhood of $B \backslash \overline{D_0}$ to which both u^e and u_0^e have harmonic continuations.

THEOREM 6.1. *There exist $\varepsilon > 0$ and $\varepsilon_2 > 0$ such that if $|u^e - u_0^e|_{3+\lambda}(N) < \varepsilon$, then $B\omega$ is a contraction in $|\omega|_{1+\lambda}(|t| = 1) < \varepsilon_2$.*

Proof. Recalling (4.11),

$$\begin{aligned} B_1\omega_1 - B_1\omega_2 &= U^e(z_0 + \omega_1) - U^e(z_0 + \omega_2) - u_{0z}^e(z_0)(\omega_1 - \omega_2) \\ &= \int_0^1 \int_0^\tau u_{zz}^e(z_0 + \omega_2 + \sigma[\omega_1 - \omega_2])(\omega_1 - \omega_2)^2 \, d\sigma \, d\tau \\ &\quad + [u_z^e(z_0 + \omega_2) - u_{0z}^e(z_0)](\omega_1 - \omega_2). \end{aligned}$$

This and properties of $C^{k+\lambda}$ norms imply, with $|\cdot|$ denoting $|\cdot|_{1+\lambda}(|t| = 1)$, that

$$\begin{aligned} |B\omega_1 - B\omega_2| &\leqslant 4\{|\omega_1 - \omega_2| \, |u^e - u_0^e|_{3+\lambda}(N) + |\omega_1 - \omega_2| \, |u_0^e|_{3+\lambda}(N) \\ &\quad + |u_z^e(z_0 + \omega_2) - u_{0z}^e(z_0)|\} \, |\omega_1 - \omega_2| \\ &\leqslant 4(\varepsilon_2 \varepsilon + \varepsilon_2 |u_0^e|_{2+\lambda}(N) + \varepsilon_0) \, |\omega_1 - \omega_2|. \end{aligned}$$

Hence, $\varepsilon_2 > 0$ chosen with regard to making $|u_z^e - u_{0z}^e| < \varepsilon_0$ sufficiently small on $|t| = 1$, and to be sufficiently smaller than $|u_0^e|_{2+\lambda}(N)$, and $\varepsilon > 0$ chosen sufficiently small imply the result.

7. Proof of Theorem 2.1

To show that the operator A in (4.13) has the trivial kernel, we employ the method in [7, pp. 183–193].

Using (4.13) and its complex conjugate to eliminate ω_* and $\overline{\omega_*}$, we obtain on $|t| = 1$

$$\begin{aligned} \phi(t) &= [\overline{u_{0z}^e(z_0(t)) \, z_0'(t) t}/u_{0z}^e(z_0(t)) \, z_0'(t) t] \, \overline{\phi(t)} + F(t) \\ F(t) &= B\omega(t) - [\overline{u_{0z}^e(z_0(t)) \, z_0'(t) t}/u_{0z}^e(z_0(t)) \, z_0'(t) t] \, \overline{B\omega(t)}. \end{aligned} \tag{7.1}$$

By moving $\phi(t)$ to the right side of (4.13) and dividing by the coefficient of $\overline{\omega_*}$, we get

$$\begin{aligned} \overline{\omega_*(t)} &= -[z_0'(t)t/\overline{z_0'(t)t}]\,\omega_*(t) + G(t) \qquad \text{on} \quad |t| = 1 \\ G(t) &= [\phi(t) - B\omega(t)]\, z_0'(t)t/\overline{u^e_{0z}(z_0(t))\, z_0'(t)t}. \end{aligned} \tag{7.2}$$

From Section 5 we see that (7.1), (7.2) are the problems of index $\lambda = -2$, $\mu = 2$, respectively, that Muskhelishvili considered in [8, Sect. 40].

Set $B\omega = 0$. Then according to [8], (7.1) has only the trivial solution $\phi(t) = 0$, which substituted in (7.2) results in linearly independent solutions $\omega(t)$ depending on three real parameters. From [8, pp. 103, 104],

$$\begin{aligned} &\omega(t) = X(t)(C_0 t^2 + C_1 t + C_2), \\ &C_0 = \bar{C}_2, \qquad C_1 \in \mathbb{R}, \end{aligned} \tag{7.3}$$

where $X(t)$ is the fundamental solution for the Hilbert problem (40.3) in [8]. The condition $\omega(0) = 0$ requires $C_2 = 0$, which eliminates two of the real parameters. Noting that a solution to (4.13) with $B\omega = 0$, $\phi = 0$ is $\omega(t) = itz_0'(t)$, which satisfies $\omega(0) = 0$ but not (2.2), we drop this solution and only the trivial kernel remains. Hence, the operator A is invertible and the result of Section 6 implies that for suitable $\varepsilon_1, \varepsilon_2 > 0$ the solution $(\phi, \omega) = A^{-1}B\omega$ to the nonhomogeneous problem, if it exists, is unique. For $|t| < 1$, let

$$u^i(z_0(t) + \omega(t)) = \tfrac{1}{4}\mathrm{Re}[\phi(t) + 4U_0^i(z_0(t)) + 4u^i_{0z}(z_0(t))\,\omega(t)],$$

so that the pair u^i, ω is unique. This uniqueness is independent of the way the imaginary constants were fixed in Section 4, since any change in these constants may be referred to $\mathrm{Im}\,\phi$.

Existence of the solution to the nonhomogeneous problem depends on the solvability of (7.1), and on the solvability of (7.2) subject to (2.2). The index of (7.1) is $\lambda = -2$, so accroding to [8, pp. 104, 105], there exists a unique solution $\phi(t)$ to (7.1) if the following condition is satisfied,

$$0 = \int_{|t|=1} \frac{F(t)}{X^+(t)}\, dt, \tag{7.4}$$

where

$$\begin{aligned} X^+(t) &= t[-\overline{u^e_{0z}(z_0(t))\, z_0'(t)t}/u^e_{0z}(z_0(t))\, z_0'(t)t]^{1/2} \exp(\Psi(t)), \\ \Psi(t) &= \frac{1}{2\pi}\int_{|t|=1} \frac{\Theta(\tau)}{\tau - t}\, d\tau - \frac{1}{4\pi}\int_{|t|=1} \frac{\Theta(\tau)}{\tau}\, d\tau, \\ \Theta(t) &= -2\arg[u^e_{0z}(z_0(t))\, z_0'(t)]. \end{aligned}$$

For (7.2), we first consider the problem without regard to (2.2). A particular solution is given by [8, (40.25)], and the general solution $\omega(t)$ is obtained by adding (7.3) to the particular solution. A glance at (40.25) and (7.3) is sufficient to show that C_2 can be chosen so that $\omega(0)=0$. To see that a certain choice of C_1 then gives an ω that satisfies (2.2), differentiate the sum of (40.25) and (7.3). Then

$$\omega'(0)=X(0)\left(\frac{1}{2\pi i}I'(0)+C_1\right),$$

where I' is the derivative of the first integral in (40.25). Using [8, p. 105], it can be shown that $I'(0)$ is a real number. We know that $\operatorname{Im} X(0)\neq 0$ from solving the homogeneous problem and getting $\omega(t)$ such that $\omega(0)=0$, $\operatorname{Im}\omega'(0)\neq 0$. Hence, C_1 can be chosen to make $\operatorname{Im}\omega'(0)=0$ for ω the solution to the nonhomogeneous problem (7.2).

So we conclude that there is uniqueness in this problem; but for existence, (7.4) must also be satisfied. Whether or not (7.4) is always satisfied has not been checked.

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References

1. H. Bellout, A. Friedman, and V. Isakov, Stability for an inverse problem in potential theory, *Trans. Amer. Math. Soc.*, to appear.
2. V. G. Cherednichenko, A problem in the conjugation of harmonic functions and its inverse, *Differentsial'nye Uravneniya* **18**, No. 4 (1982), 682–699.
3. A. Friedman and V. Isakov, On the uniqueness in the inverse conductivity problem with one measurement, *Indiana Univ. Math. J.* **38**, No. 3 (1989), 563–579.
4. V. Isakov, "Inverse Source Problems," Mathematical Surveys and Monographs, Vol. 34, Amer. Math. Soc., Providence, RI, 1990.
5. V. Isakov and J. Powell, On the inverse conductivity problem with one measurement, *Inverse Problems* **6** (1990), 311–318.
6. O. A. Ladyzhenskaya and N. N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," Academic Press, New York/London, 1969.
7. L. G. Mikhailov, "A New Class of Singular Integral Equations and its Application to Differential Equations with Singular Coefficients," Wolters–Noordhoff, Groningen, 1970.
8. N. I. Muskhelishvili, "Singular Integral Equations," Noordhoff, Groningen, 1953.